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SEMIGROUPS PRESENTED BY CONGRUENCE CLASSES OF REGULAR LANGUAGES -SURVEY-

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This is a survey on semigroups presented by congruence classes of regular languages.

1. Semigroups presented by regular congruences

Let A be a finite alphabet and $A^* = A^+ \cup \{\epsilon\}$ the set of all words over A .

A semigroup [monoid] S is presented by *regular congruence classes* if there exists a finite set A and there exists a surjective homomorphism ϕ of $A^+[A^*]$ to S such that for each word $W \in A^+[A^*]$, $\phi^{-1}(\phi(W))$ is a regular language.

Semigroups presented by regular congruence classes has nice properties (residually finiteness, having a solvable word problem). However, except finite semigroups, there are few examples of semigroups presented by regular congruence classes. In particular, the free Burnside semigroups are typical examples.

2. History of the free Burnside semigroups

Let A be a finite alphabet and $A^* = A^+ \cup \{\epsilon\}$ the set of all words over A .

Definition 1 .

$\mathbb{B}(m, n, t) = \langle x_1, \dots, x_t \mid (W^{m+n}, W^m), W \in A^+ \rangle$ is called
the Free Burnside semigroup.

$\mathbb{B}(m, n, 1)$ is a cyclic semigroup of order $m + n - 1$.

Theorem 1 . (Green and Rees, 1952)

- (1) $\mathbb{B}(m, 1, t)$ is finite \iff the Burnside group $\mathbb{G}(m, t)$ is finite.
- (2) $\mathbb{B}(2, 1, t)$ is the free band of order $\sum_{k=0}^t \binom{d}{k} \prod_{1 \leq i \leq k} (k - i + 1)^2$.

Conjecture 1 . (J. Brzozowski, 1969) For each $W \in A^+$,

$[W] = \{X \in A^+ \mid X = W \text{ in } \mathbb{B}(m, 1, t)\}$ is a regular language.

Conjecture 2 . (J. McCammond, 1991)

The conjecture for all free Burnside semigroups $\mathbb{B}(m, n, t)$.

Theorem 2 . (De Luca and Varricchio, 1990)

For $m \geq 5$ and $n \geq 2$,

the Brzozowski and McCammond's conjecture is true.

Theorem 3 . (J. McCammond, 1991)

For $m \geq 6$ and $n \geq 1$,

the Brzozowski and McCammond's conjecture is true.

Theorem 4 . (V. Guba, 1993)

For $m \geq 3$ and $n \geq 1$,

the Brzozowski and McCammond's conjecture is true.

Theorem 5 . (do Lago, 1993)

For $m = 2$ and $n = 2$,

the McCammond's conjecture is false.

Theorem 6 . (A. Plyushchenko, 2009)

The Brzozowski's conjecture holds for $\mathbb{B}(2, 1, t)$ if and only if the Brzozowski's conjecture holds for $\mathbb{B}(2, 1, t)$

3. Key points of the Guba's proof

In [4] and [5], Guba proved that the Brzozowski and McCammond's conjecture is true for $m \geq 3$ and $n \geq 1$. He used the method as follows :

Definition 2 .

$\mathbb{B}_k(m, n, t) = \langle x_1, \dots, x_t \mid (W^{m+n}, W^m), W \in A^k \rangle$ is called the Free Burnside semigroup.

Lemma 1 .

(a) Let $W(\in A^+)$ be a long k -periodic word.

Then $T^{sn}W = W$ in $\mathbb{B}(m, n, t)$.

(b) $X = VC$ is a right k -extension of V if and only if $\exists C$: the reduced form of V equals to one of VCD .

Lemma 2 . Let X, Y, Z be reduced word (no occurrence of $T^n W$).

Then $XZ = Y$ in $\mathbb{B}_k(m, n, t)$ if and only if $\exists V : Y \in VA^*$ and X is a right k - extension of V .

Definition 3 . (1) W is a long k -periodic word if $\exists X, Y : T^m \leq_{\text{subword}} XWY <_{\text{subword}} T^{m+s}$ (T : primitive word), XW is a left $k-1$ -extension of W and WY is a right $k-1$ -extension of W .

(2) $XW(WY)$ is a left k -extension of W if $\exists i_1 < \dots < i_r \leq k-1 : X = Z_1 \dots Z_r W$, each $Z_j Z_{j+1} \dots Z_r W$ is an immediate left i_j - extension of $Z_{j+1} \dots Z_r W$ ($1 \leq j \leq r$).

(3) $XW(WY)$ is an immediate left k -extension of W if $\exists C, D : W = CD$, XC is k -periodic, C is a long k -periodic word.

For any word W , construction of a finite automaton $\mathcal{A}(W)$:

States : $[X]$ (X is a right k -extension of some prefix of W) (for $\forall k \in \mathbb{N}$)

The number of the states is finite (since we have only to choose $X = VZ_1 \cdots Z_r$, each $VZ_r \cdots Z_{j+1}Z_j$ is an immediate right i_j -extension of $VZ_r \cdots Z_{j+1}$ ($1 \leq j \leq r$) and $|Z_j| \leq (m+n-1)i_j$.)

Initial States $[a]$ ($a \in A \cap \{\text{Prefixes of } W\}$), Terminal State $[W]$

Edges : $[X] \xrightarrow{a} [Xa]$ ($a \in A$)

Then the set of accepted words of $\mathcal{A}(W)$ is equal to $[W]$.

4. Rewriting systems of the free Burnside semigroups and do Lago's theorem

In [9], do Lago innovated new rewriting systems to analyze sturcture of the free Burnside semigroups.

Definition 4 . *Let R be a rewriting system over A . Then*

- (1) *A relator (l, r) of R with $|l| \geq |r|$ is stable if $\frac{|l| - |r|}{n}$ is the smallest period of r .*
- (2) *A rewriting system R is stable if every relator of R is stable.*

Theorem 7 (De Largo, 1996). *For $m \geq 3, n \geq 1$,*

- (1) *there exists a subset Σ of $\pi = \{(W^{m+n}, W^m) \mid W \in A^+\}$ which is stable and congruences generated by Σ or by π equal to each other.*
- (2) *Σ is a complete rewriting system.*
- (3) *$\mathbb{B}(m, n, t)$ is \mathcal{J} -above finite.*
- (4) *the Brzozowski and McCammond's conjecture is true for $\mathbb{B}(m, n, t)$.*

References

- [1] J. Brzozowski, Open problems about regular languages, " Formal Language Theory: Perspectives and Open Problems", edited by R.V. Book, Academic press, New York, (1980), 23-47.

- [2] J. McCammond, The solution to the word problem for the relatively free semigroups satisfying $T^a = T^{a+b}$ with $a \geq 6$, *Internat. J. Algebra and Comput.*(1),**1**(1991),1-32.
- [3] A. de Luca and S. Varricchio, One non counting regular classes, *Proc. of the 17 ICALP Int. Symp.,ed. M. S. Paterson,Lecture Notes in Comp. Sci.*, **443**,Springer Verlag,(1990),74-87.
- [4] V.S.Guba, The word problem for the relatively free Burnside semigroup satisfying $T^m = T^{m+n}$ with $m \geq 4$ or $m = 3, n = 1$, *Internat. J. Algebra and Comput*,**2**(1993),125-140.
- [5] V.S.Guba, The word problem for the relatively free Burnside semigroup satisfying $T^m = T^{m+n}$ with $m \geq 3$, *Internat. J. Algebra and Comput*,**3**(1993),335-347.
- [6] V.S.Guba, Some properties of periodic words, *Mathematical Notes*,**3**(2002),301-307.
- [7] L. Polák, On free semigroups satisfying $x^r = x$. *Simon Stevin*, **64**(1990), no. 1, 3-19.
- [8] A.P. do Lago, On free Burnside semigroups $x^n = x^{n+m}$, *Internat. J. Algebra and Comput*,**3**(1996),176-227.
- [9] A.N. Plyushchenko, Overlap-free words and free Burnside semigroup with two generators satisfying $x^2 = x^3$, *Sib. Élektron. Mat. Izv.*, **6**(2009), 166-181.